# Relations among Higher Order QCD corrections *

J. Blümlein[a], V. Ravindran[a] and W.L. van Neerven[b]

[a]DESY-Zeuthen, Platanenallee 6, D-15738 Zeuthen, Germany.

[b]Instituut-Lorentz, University of Leiden, P.O. Box 9506, 2300 HA Leiden, The Netherlands.

We study the scheme transformation of next to leading order QCD corrections to various processes. An interesting relation by Drell, Levy and Yan (DLY) among space like and time like processes is studied carefully in the next to leading order level. We construct factorisation scheme invariants and show that they are DLY–invariant.

## 1. INTRODUCTION

The Deep Inelastic Scattering (DIS) [ 1] of a lepton ($l$) on a hadron target (say proton $P$) is given by the process: $l+P(p) \rightarrow l+X$ where $p$ is the momentum of the target hadron. The $X$ in the above process denotes the final state hadrons which are summed over. The hadronic part which involves the interaction of a virtual photon of momentum $q$ with virtuality $Q^2=-q^2$ with the hadron gives information about the short distance structure of the hadron. Usually, one studies this hadronic part in terms of structure functions $F_i(Q^2, p\cdot q)$ $(i=1,L)$. From the parton model it follows that the functions only depend on the scaling variable $x_B=Q^2/2p\cdot q$, when one considers the situation where the scales $Q^2$ and $p\cdot q$ are very large [ 2]. The parton model explains the scaling behavior in terms of what are known as parton distribution functions. Due to the interaction between the partons in Quantum Chromodynamics(QCD), the scaling is violated [ 3] and hence these structure functions are no longer just functions of $x_B$ alone, but dependent on the scale $Q^2$ as well. The structure functions can be expressed in terms of these quark and gluon distribution functions with appropriate coefficients functions $C_{q,g}(z, Q^2/M^2), M^2$ being the factorisation scale.

The hadroproduction at $e^+e^-$ colliders provide a wealth of information about how unobserved partons produced in the reaction fragment into observed hadrons [ 4]. These cross sections are also prametrized in terms of what are called fragmentation functions. These functions in the QCD inspired parton model can be expressed in terms of parton fragmentation functions convoluted with parton level cross sections. The corresponding scaling variable for this process is defined as $x_E=2p\cdot q/Q^2$, where $q$, $p$ are the momenta of photon of virtuality ($Q^2=q^2$) and the produced hadron respectively.

## 2. SCHEME TRANSFORMATION

The partonic cross sections computed in perturbative QCD suffer from various divergences such as the infrared, ultraviolet and collinear singularities. For inclusive quantities all but the mass singularities cancel. The latter ones are absorbed into the parton distribution functions at a factorisation scale $M^2$. In practice, one first separates the singular part of the partonic cross sections into a process independent function, called transition function $\hat{\Gamma}(1/\epsilon, \alpha_s(R^2), M^2/\mu^2, M^2/R^2, scheme)$, which contains only the mass singularities. This object is then convoluted with the bare parton distributions and the resulting ones are called renormalized parton distributions. Hence, the perturbatively calculable coefficient functions computed in QCD suffer from scheme dependency. The scheme dependency appearing in the coefficient functions and the parton distribution functions are expected to cancel since the convolution of

*Work supported in part by EU contract FMRX-CT98-0194

them, which is the structure function, is a physical observable and is thus scheme independent [ 5]. The above discussion holds for time like processes as well.

Consider any two physical observables denoted by $F_A^N(Q^2)$ and $F_B^N(Q^2)$:

$$F_I^N(Q^2) = \int_0^1 dx x^{N-1} F_I(x,Q^2) \qquad I = A, B \quad (1)$$

For simplicity we consider the singlet case only. The general case is dealt with in Ref. [ 6]

$$F_I^N(Q^2) = f_q^{(S)N}(Q^2) C_I^{(S)N}(Q^2) + g^N(Q^2) C_I^{(S)N}(Q^2), \quad (2)$$

with

$$C_{Ii}^{(S)N}(Q^2) = \int_0^1 dz z^{N-1} C_{Ii}^{(S)}(z,Q^2), \quad (3)$$

where $i = q, g$. Using the fact that $f_q^{(S)N}(Q^2)$ and $g^N(Q^2)$ satisfy renormalization group equations (RGE), defining

$$t = -\frac{2}{\beta_0} \log\left(\frac{a_s(Q^2)}{a_s(Q_0^2)}\right), \quad (4)$$

where $a_s(Q^2) = \alpha_s(Q^2)/4\pi$, expanding the anomalous dimensions and coefficient functions in terms of the strong coupling constant one finds that

$$\frac{\partial}{\partial t}\left(\begin{array}{c} F_A^N \\ F_B^N \end{array}\right) = -\frac{1}{4}\left(\begin{array}{cc} \Gamma_{AA}^N & \Gamma_{AB}^N \\ \Gamma_{BA}^N & \Gamma_{BB}^N \end{array}\right)\left(\begin{array}{c} F_A^N \\ F_B^N \end{array}\right). \quad (5)$$

The matrix entries $\Gamma_{IJ}^N$ depend on the anomalous dimensions and the coefficient functions. Though the anomalous dimensions and coefficient functions depend on the scheme in which they are computed, the entries $\Gamma_{IJ}^N$ constructed out of them are scheme independent, because the structure functions $F_I^N(Q^2)$ are physical observables which do not depend on the factorisation scheme. This property is studied in the forthcoming sections.

In general, the functions $\Gamma_{IJ}^N$ have the following expansion in terms of coupling constant:

$$\Gamma_{ij} = \sum_{l=0}^{\infty} a_s^l(Q^2) \Gamma_{ij}^{(l)}, \quad (6)$$

where we have suppressed the $N$ dependence. Consider the choice where

$$F_A^N(Q^2) = F_2^{N(S)}(Q^2), F_B^N(Q^2) = \frac{F_L^N(Q^2)}{a_s(Q^2) C_{Lg}^{(1)}} \quad (7)$$

For convenience, $F_B^N(Q^2)$ is normalized by a factor $a_s(Q^2)C_{Lg}^{(1)}$ which is factorisation scheme independent. Since the leading order terms are made out of leading order anomalous dimensions and coefficient functions, they are scheme invariants. To next to leading order in $a_s(Q^2)$, the entries are lengthy, so we present here $\Gamma_{22}^{(1)}$ only (see Ref. [ 6] for the other entries):

$$\begin{aligned}\Gamma_{22}^{(1)} &= \gamma_{qq}^{(1)} - \frac{\beta_1}{\beta_0}\gamma_{qq}^{(0)} - \frac{C_{Lq}^{N(1)}}{C_{Lg}^{N(1)}}\Big(\gamma_{qg}^{(1)} \\ &-\frac{\beta_1}{\beta_0}\gamma_{qg}^{(0)}\Big) + \frac{C_{Lq}^{N(1)}}{C_{Lg}^{N(1)}} C_{2g}^{N(1)} \gamma_{qq}^{(0)} \\ &- \left[\frac{C_{Lq}^{N(2)}}{C_{Lg}^{N(1)}} + \left(\frac{C_{Lq}^{N(1)}}{C_{Lg}^{N(1)}}\right)^2 C_{2g}^{N(1)} \right. \\ &\left. -\frac{C_{Lq}^{N(1)}}{C_{Lg}^{N(1)}}\frac{C_{Lg}^{N(2)}}{C_{Lg}^{N(1)}}\right]\gamma_{qg}^{(0)} + C_{2g}^{N(1)}\gamma_{gq}^{(0)} \\ &-\frac{C_{Lq}^{N(1)}}{C_{Lg}^{N(1)}} C_{2g}^{N(1)}\gamma_{gg}^{(0)} + 2\beta_0\Big(C_{2q}^{N(1)} \\ &-\frac{C_{Lq}^{N(1)}}{C_{Lg}^{N(1)}} C_{2g}^{N(1)}\Big),\end{aligned} \quad (8)$$

The form of the time like $\Gamma_{IJ}$'s is same as that in the space like case but with obvious changes such as $\gamma_{ij}$ and $C_{Ij}^N$ are replaced by the corresponding time like ones. One verifies that the $\Gamma_{IJ}$'s are invariant under factorisation scheme transformations and hence they are physical observables. It is worth emphasising the fact that the scheme dependency coming from the two loop anomalous dimensions cancels exactly the scheme dependency coming from the $\mathcal{O}(a_s^2(Q^2))$ coefficient functions.

## 3. DRELL-LEVY-YAN RELATIONS

In this section we study in detail, an interesting relation between deep inelastic lepton hadron

scattering and $e^+e^-$ annihilation to a hadron plus anything else, proposed by Drell, Levy and Yan [ 4]. According to their work, if the Bjorken limit exists for both DI scattering and DI annihilation, then the scaling structure functions satisfy the following relation:

$$F_i^T(x_E) = F_i^S(1/x_B) \qquad i = 1(T), 2. \tag{9}$$

In other words, $F_i^T(x)$ are the continuations of the corresponding functions $F_i^S(x)$ from $x < 1$ to $x > 1$. This is true only when the continuation is smooth, i.e. there are no singularities, for example at $x = 1$, or others. This relation is called DLY relation in the literature.

In this section, we study this property in more detail [ 6, 7]. As we know, the splitting functions and coefficient functions are not physical due to the scheme choice to renormalise mass singularities. Hence, the naive continuation rule is violated in general. It was demonstrated in the paper by Curci, Furmanski and Petronzio [ 8] that by appropriately modifying the continuation rule in the $\overline{\text{MS}}$ scheme, one can show that the time like splitting functions are related to space like counter parts. Since the modification of the continuation rule is related to the scheme one adopts, it simply amounts to finding finite renormalisation factors for these quantities. It was shown that the finite renormalisation factors can be constructed from the $\epsilon$–dependent part of the splitting function when computed in dimensional regularisation [ 9]. In addition to this, care should be taken when dealing with quarks and gluonic states which was not the case yet in the DLY work. It amounts to multiplying by $-1$ for continued space like $P_{qq}, P_{gg}$, $C_F/(2n_fT_f)$ for $P_{qg}$, and $2n_fT_f/C_F$ for $P_{gq}$. This is independent of the scheme because it results from the crossing of the particles between *in* and *out* states. Keeping this in mind and using the known splitting functions [ 8, 10] one finds that

$$\begin{aligned} P_{ij} &= \sum_{\{k,l\}=q,g} Z_{ik}^T \otimes \bar{P}_{kl} \otimes (Z^{T-1})_{lj} \\ &\quad -2\beta(a_s) \sum_{l=q,g} Z_{il}^T \otimes \frac{d}{da_s}(Z^{T-1})_{lj}, \end{aligned} \tag{10}$$

where the quantities with a bar on the top denote that they are continued from $z \to 1/z$ with appropriate factors in front. The relations given above remain true for the polarized splitting functions [ 11, 9] as well. The renormalisation factors are given by

$$Z_{ij}^T \quad = \quad P_{ji}^{(0)}\Big(\log(z) + a_{ji}\Big). \tag{11}$$

The terms $a_{ij}$ depend on whether polarized or unpolarized splitting functions are considered. In the case of unpolarized splitting functions, one finds that

$$a_{qq} = a_{gg} = 0, \quad a_{qg} = -1/2, \quad a_{gq} = 1/2. \tag{12}$$

For the polarized case, one obtains

$$a_{ij} = 0. \tag{13}$$

The log in the renormalisation factors originates from the kinematics and the factor $\pm 1/2$ come from the polarisation averaging of the gluons.

At this point it is worth comparing with the work of Gribov and Lipatov [ 12] (GL) in which $P_{qq}^{(S)}(z) = P_{qq}^{(T)}(z)$ is claimed. This relation is preserved at the leading order in $a_s$ but broken at higher orders. Using the method of [ 8], the breaking terms can be identified with those coming from the ladder diagrams beyond leading order. Ref. [ 8] shows that the GL relation is broken even for the physical quantities unlike the DLY-relation by considering scheme invariant combination of non–singlet structure functions. In this section we will substantiate their result by looking at the singlet scheme invariant physical quantities.

Now, let us study how space like and time like coefficient functions are related. The coefficient functions are nothing but the parton level cross sections renormalized by mass factorisation. Hence it is expected to violate the DLY relation due to the scheme in which they are computed. The leading order longitudinal coefficient functions are identically zero. The next to leading order longitudinal ones do not get any correction. The reason for this is that the unrenormalised longitudinal coefficient functions do not contain any mass singularities unlike the transverse coefficient functions, hence there is no left over finite

piece which could arise from the $z^\epsilon$ terms or the $n-$diminsional polarisation average. At NNLO level, the longitudinal coefficient functions alone do not satisfy the DLY relation anywhere. We follow the results given in [ 13, 14, 15] for the space like and [ 16, 17] for the time like case. It turns out that they are related by the $Z$ factors in a non–linear way as given below.

$$
\begin{aligned}
C_{Lq}^{(2)T}(z) - \frac{z}{2} C_{Lq}^{(2)S}\left(\frac{1}{z}\right) &= Z_{qq}^T \otimes \frac{z}{2} C_{Lq}^{(1)S}\left(\frac{1}{z}\right) \\
&+ Z_{gq}^T \otimes \frac{C_F}{2n_f T_f} \frac{-z}{2} C_{Lg}^{(1)S}\left(\frac{1}{z}\right) \\
\frac{1}{2}\left[C_{Lg}^{(2)T}(z) - \frac{C_F}{2n_f T_f}\frac{z}{2} 2C_{Lg}^{(2)S}\left(\frac{1}{z}\right)\right] & \\
&= Z_{qg}^T \otimes \frac{z}{2} C_{Lq}^{(1)S}\left(\frac{1}{z}\right) \\
&+ Z_{gg}^T \otimes \frac{C_F}{2n_f T_f} \frac{-z}{2} C_{Lg}^{(1)S}\left(\frac{1}{z}\right). \qquad (14)
\end{aligned}
$$

The right hand side of the above equation contains the various convolutions of $Z$ factors with the continued NLO longitudinal space like coefficient functions.

Let us define the difference between time like quantities $(\Gamma_{ij}^T)$and continued space like quantities$(\Gamma_{ij}^S)$ as $\delta\Gamma_{ij} = \Gamma_{ij}^T - \bar{\Gamma}_{ij}^S$. Using the $N$th moment of Eq. (14) , we get

$$
\begin{aligned}
\delta\Gamma_{22}^{(1)} \;=\; & \delta\gamma_{qq}^{(1)} - 2\beta_0 Z_{qq}^{NT} - \bar{\gamma}_{gq}^{(0)} Z_{qg}^{NT} \\
& + \bar{\gamma}_{qg}^{(0)} Z_{gq}^{NT} + \frac{\bar{C}_{Lq}^{N(1)}}{\bar{C}_{Lg}^{N(1)}}\Big( - \delta\gamma_{gq}^{(1)} \\
& + 2\beta_0 Z_{qg}^{NT} - Z_{qg}^{NT}\bar{\gamma}_{qq}^{(0)} + Z_{qq}^{NT}\bar{\gamma}_{qg}^{(0)} \\
& - Z_{gg}^{NT}\bar{\gamma}_{qg}^{(0)} + Z_{qg}^{NT}\bar{\gamma}_{gg}^{(0)}\Big). \qquad (15)
\end{aligned}
$$

Eq. (10) implies that $\delta\Gamma_{22}^{(1)} = 0$. The same is true for other entries of the $\Gamma$ matrix. It is clear from the above exercise that for time like physical anomalous dimensions $\Gamma_{ij}^T$, one can directly use the space like physical anomalous dimension with the appropriate changes such as $z \to 1/z$ and the corresponding changes in the overall colour factors *without* using any $Z$ factors. Also, the GL relation cannot be thought of as scheme transformation, hence the physical anomalous dimensions are not preserved in this case.

## REFERENCES


1. J. Blümlein, J. Huston, C. Royon, R. Yoshida, DESY-97-121, June 1997, 5th International Workshop on Deep Inelastic Scattering and QCD (DIS 97), Chicago, IL, 14-18 Apr 1997, hep-ph/9707420.
2. J. D. Bjorken, Phys. Rev. **179** (1969) 1547.
3. D.J. Gross and F. Wilczek, Phys. Rev. Lett. **30** (1973) 1343; H.D. Politzer, Phys. Rev. Lett. **30** (1973) 1346; S. Coleman and D.J. Gross, Phys. Rev. Lett. **31** (1973) 851.
4. S.D. Drell, D.J. Levy and T.M. Yan, Phys. Rev. **D1** (1970) 1617.
5. W. Furmanski and R. Petronzio, Z. Phys. **C11** (1982) 293.
6. J. Blümlein, V. Ravindran and W.L. van Neerven, DESY-98/144.
7. J. Blümlein, V. Ravindran and W.L. van Neerven, Acta Phys. Polon. **B29** (1998) 2581.
8. G. Curci, W. Furmanski and R. Petronzio, Nucl. Phys. **B175** (1980) 27; E.G. Floratos, R. Lacaze and C. Kounnas, Phys. Lett. **B98** (1981) 89,285; Nucl. Phys. **B192** (1981) 417.
9. M. Stratmann and W. Vogelsang, Nucl. Phys. **B496** (1997) 41.
10. W. Furmanski and R. Petronzio, Phys. Lett. **B97** (1980) 437.
11. R. Mertig and W.L. van Neerven, Z. Phys. **C70** (1996) 637, W. Vogelsang, Phys. Rev. **D54** (1996) 2023, Nucl. Phys. **B475** (1996) 47.
12. V.N. Gribov, L.N. Lipatov, Sov. J. Nucl. Phys. **15** (1972) 675.
13. J. Sanchez Guille'n et al., Nucl. Phys. **B353** (1991) 337.
14. W.L. van Neerven and E.B. Zijlstra, Phys. Lett. **B272** (1991) 127, Phys. Lett. **B273** (1991) 476; Nucl. Phys. **B383** (1992) 525, Phys. Lett. **B297** (1992) 377; J. Blümlein, S. Kurth, DESY–97/160, hep-ph/9708388; DESY–98/141, hep-ph/9810241, Phys. Rev. **D** in print.
15. E. B. Zijlstra, Ph.D thesis (1993).
16. P.J. Rijken and W.L. van Neerven, Nucl. Phys. **523** (1998) 245.
17. P.J. Rijken, Ph.D thesis (1997).